

RELATIONS AND FUNCTIONS

▶ A relation R from a set A to a set B is a subset of the cartesian product $A \times B$ obtained by describing a relationship between the first element x and the second element y of the ordered pairs in $A \times B$.

▶ **Function** : A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B. We write

$f: A \rightarrow B$, where $f(x) = y$.

▶ A function $f: X \rightarrow Y$ is one-one (or injective) if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X.$$

▶ A function $f: X \rightarrow Y$ is onto (or surjective) if given any $y \in Y$, $\exists x \in X$ such that $f(x) = y$.

▶ **Many-One Function** :

A function $f: A \rightarrow B$ is called many-one, if two or more different elements of A have the same f-image in B.

▶ **Into function** :

A function $f: A \rightarrow B$ is into if there exist at least one element in B which is not the f-image of any element in A.

▶ **Many One-Onto function** :

A function $f: A \rightarrow R$ is said to be many one-onto if f is onto but not one-one.

▶ **Many One-Into function** :

A function is said to be many one-into if it is neither one-one nor onto.

▶ A function $f: X \rightarrow Y$ is invertible if and only if f is one-one and onto.

TRIGONOMETRIC FUNCTIONS AND EQUATIONS

▶ **General Solution of the equation**

$\sin \theta = 0$:

when $\sin \theta = 0$

$$\theta = n\pi; n \in I \text{ i.e. } n = 0, \pm 1, \pm 2, \dots$$

General solution of the equation

$\cos \theta = 0$:

when $\cos \theta = 0$

$$\theta = (2n+1)\pi/2, n \in I \text{ i.e. } n = 0, \pm 1, \pm 2, \dots$$

General solution of the equation $\tan \theta = 0$:

General solution of $\tan \theta = 0$ is $\theta = n\pi; n \in I$

▶ **General solution of the equation**

(a) **$\sin \theta = \sin \alpha$:** $\theta = n\pi + (-1)^n \alpha; n \in I$

(b) **$\sin \theta = k$, where $-1 \leq k \leq 1$.**

$$\theta = n\pi + (-1)^n \alpha, \text{ where } n \in I \text{ and } \alpha = \sin^{-1} k$$

(c) **$\cos \theta = \cos \alpha$:** $\theta = 2n\pi \pm \alpha, n \in I$

(d) **$\cos \theta = k$, where $-1 \leq k \leq 1$.**

$$\theta = 2n\pi \pm \alpha, \text{ where } n \in I \text{ and } \alpha = \cos^{-1} k$$

(e) **$\tan \theta = \tan \alpha$:** $\theta = n\pi + \alpha; n \in I$

(f) **$\tan \theta = k$,** $\theta = n\pi + \alpha$, where $n \in I$ and $\alpha = \tan^{-1} k$

(g) **$\sin^2 \theta = \sin^2 \alpha$:** $\theta = n\pi \pm \alpha; n \in I$

(h) **$\cos^2 \theta = \cos^2 \alpha$:** $\theta = n\pi \pm \alpha; n \in I$

(i) **$\tan^2 \theta = \tan^2 \alpha$:** $\theta = n\pi \pm \alpha; n \in I$

▶ $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots$ to n terms

$$= \frac{\sin \left[\alpha + \left(\frac{n-1}{2} \right) \beta \right] \left[\sin \left(\frac{n\beta}{2} \right) \right]}{\sin (\beta / 2)} ; \beta \neq 2n\pi$$

▶ $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots$ to n terms

$$= \frac{\cos \left[\alpha + \left(\frac{n-1}{2} \right) \beta \right] \left[\sin \left(\frac{n\beta}{2} \right) \right]}{\sin \left(\frac{\beta}{2} \right)} ; \beta \neq 2n\pi$$

▶ $\tan \left(\frac{B-C}{2} \right) = \left(\frac{b-c}{b+c} \right) \cot \left(\frac{A}{2} \right)$

▶ $\sin \left(\frac{A}{2} \right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$

▶ $\tan \left(\frac{A}{2} \right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

▶ $R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$

▶ $R = \frac{abc}{4\Delta}$

▶ $r = 4R \sin \left(\frac{A}{2} \right) \cdot \sin \left(\frac{B}{2} \right) \cdot \sin \left(\frac{C}{2} \right)$

▶ $a = c \cos B + b \cos C$

▶ Maximum value of $a \sin \theta + b \cos \theta = \sqrt{a^2 + b^2}$ and minimum value of $a \sin \theta + b \cos \theta = -\sqrt{a^2 + b^2}$

INVERSE TRIGONOMETRIC FUNCTIONS

▶ **Properties of inverse trigonometric function**

• $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } xy < 1 \\ \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x > 0, y > 0 \\ & \text{and } xy > 1 \\ -\pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x < 0, y < 0 \\ & \text{and } xy > 1 \end{cases}$$

• $\tan^{-1} x - \tan^{-1} y$

$$= \begin{cases} \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } xy > -1 \\ \pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

• $\sin^{-1} x + \sin^{-1} y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), & \text{if } -1 \leq x, y \leq 1 \text{ and } x^2 + y^2 \leq 1 \\ & \text{or if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), & \text{if } 0 < x, y \leq 1 \\ & \text{and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), & \text{if } -1 \leq x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

• $\cos^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}), & \text{if } -1 \leq x, y \leq 1 \text{ and } x + y \geq 0 \\ 2\pi - \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}), & \text{if } -1 \leq x, y \leq 1 \text{ and } x + y < 0 \end{cases}$$

$$2 \sin^{-1} x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt{2}} \leq x \leq 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1 \leq x \leq -\frac{1}{\sqrt{2}} \end{cases}$$

$$2 \tan^{-1} x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x < -1 \end{cases}$$

QUADRATIC EQUATIONS AND INEQUALITIES

• **Roots of a Quadratic Equation :** The roots of the quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Nature of roots : In Quadratic equation $ax^2 + bx + c = 0$. The term $b^2 - 4ac$ is called discriminant of the equation. It is denoted by Δ or D .

(A) Suppose $a, b, c \in \mathbb{R}$ and $a \neq 0$

- (i) If $D > 0 \Rightarrow$ Roots are Real and unequal
- (ii) If $D = 0 \Rightarrow$ Roots are Real and equal and each equal to $-b/2a$
- (iii) If $D < 0 \Rightarrow$ Roots are imaginary and unequal or complex conjugate.

(B) Suppose $a, b, c \in \mathbb{Q}$ and $a \neq 0$

- (i) If $D > 0$ and D is perfect square \Rightarrow Roots are unequal and Rational
- (ii) If $D > 0$ and D is not perfect square \Rightarrow Roots are irrational and unequal.

• **Condition for Common Root(s)**

Let $ax^2 + bx + c = 0$ and $dx^2 + ex + f = 0$ have a common root α (say).

Condition for both the roots to be common is $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

- If $p + iq$ (p and q being real) is a root of the quadratic equation, where $i = \sqrt{-1}$, then $p - iq$ is also a root of the quadratic equation.
- Every equation of n^{th} degree ($n \geq 1$) has exactly n roots and if the equation has more than n roots, it is an identity.

COMPLEX NUMBERS

• **Exponential Form:** If $z = x + iy$ is a complex number then its exponential form is $z = re^{i\theta}$ where r is modulus and θ is amplitude of complex number.

• (i) $|z_1| + |z_2| \geq |z_1 + z_2|$; here equality holds when $\arg(z_1/z_2) = 0$ i.e. z_1 and z_2 are parallel.

(ii) $\left| |z_1| - |z_2| \right| \leq |z_1 - z_2|$; here equality holds when $\arg(z_1/z_2) = \pi$ i.e. z_1 and z_2 are parallel.

(iii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

• $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

• $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

• For any integer $k, i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$

• $|z - z_1| + |z - z_2| = \lambda$, represents an ellipse if $|z_1 - z_2| < \lambda$, having the points z_1 and z_2 as its foci. And if $|z_1 - z_2| = \lambda$, then z lies on a line segment connecting z_1 and z_2 .

• **Properties of Cube Roots of Unity**

- (i) $1 + \omega + \omega^2 = 0$ (ii) $\omega^3 = 1$
- (iii) $1 + \omega^n + \omega^{2n} = 3$ (if n is multiple of 3)
- (iv) $1 + \omega^n + \omega^{2n} = 0$ (if n is not a multiple of 3).

PERMUTATIONS AND COMBINATIONS

• The number of permutations of n different things, taken r at a time, where repetition is allowed, is n^r .

• **Selection of Objects with Repetition :** The total number of selections of r things from n different things when each thing may be repeated any number of times is ${}^{n+r-1}C_r$

• **Selection from distinct objects :** The number of ways (or combinations) of n different things selecting at least one of them is ${}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combination of n different things.

• **Selection from identical objects :** The number of ways to select some or all out of $(p + q + r)$ things where p are alike of first kind, q are alike of second kind and r are alike of third kind is $(p + 1)(q + 1)(r + 1) - 1$

• **Selection when both identical and distinct objects are present:** If out of $(p + q + r + t)$ things, p are alike one kind, q are alike of second kind, r are alike of third kind and t are different, then the total number of combinations is $(p + 1)(q + 1)(r + 1)2^t - 1$

• **Circular permutations:**
(a) Arrangements round a circular table : The number of circular permutations of n different things taken all at a time is $\frac{{}^n P_n}{n} = (n - 1)!$, if clockwise and anticlockwise orders are taken as different.

(b) **Arrangements of beads or flowers (all different) around a circular necklace or garland:**

The number of circular permutations of 'n' different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken to be some.

Sum of numbers :

(a) For given n different digits $a_1, a_2, a_3, \dots, a_n$ the sum of the digits in the unit place of all numbers formed (if numbers are not repeated) is $(a_1 + a_2 + a_3 + \dots + a_n)(n-1)!$

(b) Sum of the total numbers which can be formed with given n different digits a_1, a_2, \dots, a_n is $(a_1 + a_2 + a_3 + \dots + a_n)(n-1)!$. (111n times)

BINOMIAL THEOREM

Greatest binomial coefficients : In a binomial expansion binomial coefficients of the middle terms are called as greatest binomial coefficients.

(a) If n is even : When $r = \frac{n}{2}$ i.e. ${}^nC_{n/2}$ takes maximum value.

(b) If n is odd : $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

i.e. ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$ and take maximum value.

Important Expansions :

If $|x| < 1$ and $n \in \mathbb{Q}$ but $n \notin \mathbb{N}$, then

(a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots$

(b) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}(-x)^r + \dots$

SEQUENCE AND SERIES

Properties related to A.P. :

(i) Common difference of AP is given by $d = S_2 - 2S_1$ where S_2 is sum of first two terms and S_1 is sum of first term.

(ii) If for an AP sum of p terms is q, sum of q terms is p, then sum of (p+q) term is (p+q).

(iii) In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.

(iv) If terms $a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{2n+1}$ are in A.P., then sum of these terms will be equal to $(2n+1)a_{n+1}$.

(v) If for an A.P. sum of p terms is equal to sum of q terms then sum of (p+q) terms is zero

(vi) Sum of n AM's inserted between a and b is equal to n

times the single AM between a and b i.e. $\sum_{r=1}^n A_r = nA$

where $A = \frac{a+b}{2}$

The geometric mean (G.M.) of any two positive numbers a and b is given by \sqrt{ab} i.e., the sequence a, G, b is G.P.

n GM's between two given numbers: If in between two numbers 'a' and 'b', we have to insert n GM G_1, G_2, \dots, G_n then $a, G_1, G_2, \dots, G_n, b$ will be in G.P.

The series consist of (n+2) terms and the last term is b and first term is a.

$\Rightarrow ar^{n+2-1} = b \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$G_1 = ar, G_2 = ar^2, \dots, G_n = ar^n$ or $G_n = b/r$

Use of inequalities in progression :

(a) Arithmetic Mean \geq Geometric Mean

(b) Geometric Mean \geq Harmonic Mean :

$A \geq G \geq H$

STRAIGHT LINES

An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by

$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|, 1 + m_1 m_2 \neq 0$

Three points A, B and C are collinear, if and only if slope of AB = slope of BC.

The equation of the line having normal distance from origin is p and angle between normal and the positive x-axis is ω , is given by $x \cos \omega + y \sin \omega = p$.

Co-ordinate of some particular points :

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

Incentre : Co-ordinates of incentre

$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$

where a, b, c are the sides of triangle ABC

Area of a triangle : Let $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) respectively be the coordinates of the vertices A, B, C of a triangle ABC. Then the area of triangle ABC, is

$\frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Or

$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

CONIC SECTIONS

➤ **Condition of Tangency : Circle**
 $x^2 + y^2 = a^2$ will touch the line.

$y = mx + c$ if $c = \pm a\sqrt{1+m^2}$

➤ **Pair of Tangents :** From a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$. Their combined equation is $SS_1 = T^2$.

➤ **Condition of Orthogonality :** If the angle of intersection of the two circle is a right angle ($\theta = 90^\circ$) then such circle are called Orthogonal circle and conditions for their orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

➤ **Tangent to the parabola :**

Condition of Tangency : If the line $y = mx + c$ touches a parabola $y^2 = 4ax$ then $c = a/m$

➤ **Tangent to the Ellipse:**

Condition of tangency and point of contact :

The condition for the line $y = mx + c$ to be a tangent to the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates

of the points of contact are $\left(\pm \frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2m^2 + b^2}} \right)$

➤ **Normal to the ellipse**

(i) **Point Form :** The equation of the normal to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

(ii) **Parametric Form :** The equation of the normal to the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos\theta, b \sin\theta)$ is

$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$

➤ **Tangent to the hyperbola :**

Condition for tangency and points of contact : The condition for the line $y = mx + c$ to be a tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the

points of contact are $\left(\pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2m^2 - b^2}} \right)$

➤ **Chord of contact :**

The equation of chord of contact of tangent drawn from a

point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T = 0$

where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

➤ **Equation of normal in different forms :**

Point Form : The equation of the normal to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

THREE DIMENSIONAL GEOMETRY

➤ **Slope Form :** The equation of normal to

the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$

➤ **Conditions of Parallelism and Perpendicularity of Two Lines:**

Case-I : When dc's of two lines AB and CD, say ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are known.

$AB \parallel CD \Leftrightarrow \ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$

$AB \perp CD \Leftrightarrow \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

Case-II : When dr's of two lines AB and CD, say a_1, b_1, c_1 and a_2, b_2, c_2 are known

$AB \parallel CD \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$AB \perp CD \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

➤ If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then $\cos \theta = |\ell_1\ell_2 + m_1m_2 + n_1n_2|$.

➤ Equation of a line through a point (x_1, y_1, z_1) and having

direction cosines ℓ, m, n is $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

➤ Shortest distance between $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

is $\frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$

➤ Let the two lines be

$\frac{x-\alpha_1}{\ell_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$ (1)

and $\frac{x-\alpha_2}{\ell_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$ (2)

These lines will coplanar if

$\begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

The plane containing the two lines is

$\begin{vmatrix} x - \alpha_1 & y - \beta_1 & z - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

➤ The equation of a plane through a point whose position vector is \vec{a} and perpendicular to the vector \vec{N} is

$(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

- ▶ Vector equation of a plane that passes through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.

- ▶ Two planes $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are coplanar if $(\vec{a}_2 - \vec{a}_1) + (\vec{b}_1 \times \vec{b}_2) = 0$

LIMIT

DIFFERENTIAL CALCULUS

▶ **Existence of Limit :**

$\lim_{x \rightarrow a} f(x)$ exists $\Rightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \ell$
Where ℓ is called the limit of the function

- ▶ (i) If $f(x) \leq g(x)$ for every x in the deleted nbd of a , then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$

- (ii) If $f(x) \leq g(x) \leq h(x)$ for every x in the deleted nbd of a and $\lim_{x \rightarrow a} f(x) = \ell = \lim_{x \rightarrow a} h(x)$ then $\lim_{x \rightarrow a} g(x) = \ell$

- (iii) $\lim_{x \rightarrow a} f \circ g(x) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(m)$ where $\lim_{x \rightarrow a} g(x) = m$

- (iv) If $\lim_{x \rightarrow a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \rightarrow a} \frac{1}{f(x)} = 0$

CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS

- ▶ A function $f(x)$ is said to be continuous at a point $x = a$ if

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a)$$

- ▶ **Discontinuous Functions :**

(a) Removable Discontinuity:

A function f is said to have removable discontinuity at $x = a$ if $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$ but their common value is not equal to $f(a)$.

- (b) Discontinuity of the first kind:** A function f is said to have a discontinuity of the first kind at $x = a$ if $\lim_{x \rightarrow a^-} f(x)$ and

$$\lim_{x \rightarrow a^+} f(x)$$
 both exist but are not equal.

- (c) Discontinuity of second kind:** A function f is said to have a discontinuity of the second kind at $x = a$ if neither

$$\lim_{x \rightarrow a^-} f(x) \text{ nor } \lim_{x \rightarrow a^+} f(x) \text{ exists.}$$

Similarly, if $\lim_{x \rightarrow a^+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at $x = a$.

- ▶ **For a function f :**

Differentiability \Rightarrow Continuity;

Continuity \nRightarrow derivability

Not derivability \nRightarrow discontinuous ;

But discontinuity \Rightarrow Non derivability

▶ **Differentiation of infinite series:**

- (i) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}}$

$$\Rightarrow y = \sqrt{f(x) + y} \Rightarrow y^2 = f(x) + y$$

$$2y \frac{dy}{dx} = f'(x) + \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

- (ii) If $y = f(x)^{f(x)^{f(x)^{\dots \infty}}}$ then $y = f(x)^y$.

$$\therefore \log y = y \log [f(x)]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{y' \cdot f'(x)}{f(x)} + \log f(x) \cdot \left(\frac{dy}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y \log f(x)]}$$

- (iii) If $y = f(x) + \frac{1}{f(x) + \frac{1}{f(x) + \frac{1}{f(x)} \dots \dots \dots}$ then

$$\frac{dy}{dx} = \frac{y f'(x)}{2y - f(x)}$$

DIFFERENTIATION AND APPLICATION

- ▶ **Interpretation of the Derivative :** If $y = f(x)$ then, $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$

▶ **Increasing/Decreasing :**

- (i) If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I .

- (ii) If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I .

- (iii) If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I .

▶ **Test of Local Maxima and Minima –**

First Derivative Test – Let f be a differentiable function defined on an open interval I and $c \in I$ be any point. f has a local maxima or a local minima at $x = c$, $f'(c) = 0$.

$$\text{Put } \frac{dy}{dx} = 0 \text{ and solve this equation for } x. \text{ Let } c_1, c_2, \dots, c_n$$

be the roots of this.

If $\frac{dy}{dx}$ changes sign from +ve to -ve as x increases

through c_1 then the function attains a local max at $x = c_1$

If $\frac{dy}{dx}$ changes its sign from -ve to +ve as x increases

through c_1 then the function attains a local minimum at $x = c_1$

If $\frac{dy}{dx}$ does not changes sign as increases through c_1

then $x = c_1$ is neither a point of local max^m nor a point of local min^m. In this case x is a point of inflexion.

▶ **Rate of change of variable :**

The value of $\frac{dy}{dx}$ at $x = x_0$ i.e. $\left(\frac{dy}{dx}\right)_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$

If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided that $\frac{dx}{dt} \neq 0$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x each with respect to t .

▶ Length of Sub-tangent = $\left|y \frac{dx}{dy}\right|$; Sub-normal = $\left|y \frac{dy}{dx}\right|$;

$$\text{Length of tangent} = \left|y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right|$$

$$\text{Length of normal} = \left|y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right|$$

▶ **Equations of tangent and normal :** The equation of the tangent at $P(x_1, y_1)$ to the curve $y = f(x)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_p (x - x_1)$$

The equation of the normal at $P(x_1, y_1)$ to the curve $y = f(x)$ is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_p} (x - x_1)$$

INTEGRAL CALCULUS

▶ **Two standard forms of integral :**

$$\begin{aligned} \int e^x [f(x) + f'(x)] dx &= e^x f(x) + c \\ \Rightarrow \int e^x [f(x) + f'(x)] dx &= \int e^x f(x) dx + \int e^x f'(x) dx \\ &= e^x f(x) - \int e^x f'(x) dx + \int e^x f(x) dx \end{aligned}$$

(on integrating by parts) = $e^x f(x) + c$

▶ Table shows the partial fractions corresponding to different type of rational functions :

S. No.	Form of rational function	Form of partial fraction
1.	$\frac{px + q}{(x - a)(x - b)}$	$\frac{A}{(x - a)} + \frac{B}{(x - b)}$
2.	$\frac{px^2 + qx + r}{(x - a)^2(x - b)}$	$\frac{A}{(x - a)} + \frac{B}{(x - a)^2} + \frac{C}{(x - b)}$
3.	$\frac{px^2 + qx + r}{(x - a)(x^2 + bx + c)}$	$\frac{A}{(x - a)} + \frac{Bx + C}{x^2 + bx + c}$

▶ Leibnitz rule : $\frac{d}{dx} \int_{f(x)}^{g(x)} F(t) dt = g'(x)F(g(x)) - f'(x)F(f(x))$

▶ If a series can be put in the form

$$\frac{1}{n} \sum_{r=0}^{r=n-1} f\left(\frac{r}{n}\right) \text{ or } \frac{1}{n} \sum_{r=1}^{r=n} f\left(\frac{r}{n}\right), \text{ then its limit as } n \rightarrow \infty$$

$$\text{is } \int_0^1 f(x) dx$$

▶ Area between curves :

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}] dx$$

$$\text{and } x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}] dy$$

If the curves intersect then the area of each portion must be found individually.

▶ **Symmetrical area :** If the curve is symmetrical about a coordinate axis (or a line or origin), then we find the area of one symmetrical portion and multiply it by the number of symmetrical portion to get the required area.

▶ **Probability of an event:** For a finite sample space with equally likely outcomes Probability of an event is

PROBABILITY

$$P(A) = \frac{n(A)}{n(S)}, \text{ where } n(A) = \text{number of elements in the set } A, n(S) = \text{number of elements in the set } S.$$

▶ **Theorem of total probability :** Let $\{E_1, E_2, \dots, E_n\}$ be a partition of a sample space and suppose that each of E_1, E_2, \dots, E_n has nonzero probability. Let A be any event associated with S , then $P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \dots + P(E_n)P(A|E_n)$

▶ **Bayes' theorem:** If E_1, E_2, \dots, E_n are events which constitute a partition of sample space S , i.e. E_1, E_2, \dots, E_n are pairwise disjoint and $E_1 \cup E_2 \cup \dots \cup E_n = S$ and A be any event with nonzero probability, then

$$P(E_i | A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^n P(E_j)P(A|E_j)}$$

▶ Let X be a random variable whose possible values $x_1, x_2, x_3, \dots, x_n$ occur with probabilities $p_1, p_2, p_3, \dots, p_n$ respectively.

The mean of X , denoted by μ , is the number $\sum_{i=1}^n x_i p_i$

The mean of a random variable X is also called the expectation of X , denoted by $E(X)$.

- Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(a) There should be a finite number of trials. (b) The trials should be independent. (c) Each trial has exactly two outcomes : success or failure. (d) The probability of success remains the same in each trial.

For Binomial distribution $B(n, p)$,

$$P(X = x) = {}^n C_x q^{n-x} p^x, x = 0, 1, \dots, n \quad (q = 1 - p)$$

MATRICES

- Properties of Transpose**

- (i) $(A^T)^T = A$
- (ii) $(A \pm B)^T = A^T \pm B^T$
- (iii) $(AB)^T = B^T A^T$ (iv) $(kA)^T = k(A)^T$
- (v) $I^T = I$ (vi) $\text{tr}(A) = \text{tr}(A)^T$
- (vii) $(A_1 A_2 A_3 \dots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \dots A_3^T A_2^T A_1^T$

- Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called symmetric matrix if

$$a_{ij} = a_{ji} \text{ for all } i, j \text{ or } A^T = A$$

- Skew-Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if

$$a_{ij} = -a_{ji} \text{ for all } i, j \text{ or } A^T = -A$$

Also every square matrix A can be uniquely expressed as a sum of a symmetric and skew-symmetric matrix.

- Differentiation of a matrix :** If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & \ell(x) \end{bmatrix}$ then

$$\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & \ell'(x) \end{bmatrix} \text{ is a differentiation of Matrix } A.$$

DETERMINANTS

- Properties of adjoint matrix :** If A, B are square matrices of order n and I_n is corresponding unit matrix, then

- (i) $A(\text{adj } A) = |A| I_n = (\text{adj } A)A$
- (ii) $|\text{adj } A| = |A|^{n-1}$ (Thus $A(\text{adj } A)$ is always a scalar matrix)
- (iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$
- (iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

- (v) $\text{adj}(A^T) = (\text{adj } A)^T$
- (vi) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- (vii) $\text{adj}(A^m) = (\text{adj } A)^m, m \in \mathbb{N}$
- (viii) $\text{adj}(kA) = k^{n-1}(\text{adj } A), k \in \mathbb{R}$
- (ix) $\text{adj}(I_n) = I_n$

- Properties of Inverse Matrix :** Let A and B are two invertible matrices of the same order, then

- (i) $(A^T)^{-1} = (A^{-1})^T$
- (ii) $(AB)^{-1} = B^{-1}A^{-1}$
- (iii) $(A^k)^{-1} = (A^{-1})^k, k \in \mathbb{N}$
- (iv) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$
- (v) $(A^{-1})^{-1} = A$

$$(vi) |A^{-1}| = \frac{1}{|A|} = |A|^{-1}$$

- (vii) If $A = \text{diag}(a_1, a_2, \dots, a_n)$, then $A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, \dots, a_n^{-1})$

(viii) A is symmetric matrix $\Rightarrow A^{-1}$ is symmetric matrix.

- Rank of a Matrix :** A number r is said to be the rank of a $m \times n$ matrix A if

(a) Every square sub matrix of order $(r + 1)$ or more is singular and (b) There exists at least one square submatrix of order r which is non-singular.

Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

- Using Cramer's rule of determinant we get

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta} \text{ i. e. } x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

Case-I : If $\Delta \neq 0$

$$\text{Then } x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

\therefore The system is consistent and has unique solutions.

Case-II if $\Delta = 0$ and

- (i) If at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero then the system of equations is inconsistent i.e. has no solution.
- (ii) If $d_1 = d_2 = d_3 = 0$ or $\Delta_1, \Delta_2, \Delta_3$ are all zero then the system of equations has infinitely many solutions.

VECTOR ALGEBRA

- Given vectors $x_1 \vec{a} + y_1 \vec{b} + z_1 \vec{c}$, $x_2 \vec{a} + y_2 \vec{b} + z_2 \vec{c}$, $x_3 \vec{a} + y_3 \vec{b} + z_3 \vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, will be

$$\text{coplanar if and only if } \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

- Scalar triple product :**

- (a) If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ and $\vec{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$ then

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a} \ \vec{b} \ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- (b) $[a \ b \ c]$ = volume of the parallelepiped whose coterminous edges are formed by $\vec{a}, \vec{b}, \vec{c}$
- (c) $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $[\vec{a} \ \vec{b} \ \vec{c}] = 0$

- (d) Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if

$$[\vec{AB} \ \vec{AC} \ \vec{AD}] = 0 \text{ i.e. if and only if}$$

$$[\vec{b} - \vec{a} \ \vec{c} - \vec{a} \ \vec{d} - \vec{a}] = 0$$

- (e) Volume of a tetrahedron with three coterminous edges

$$\vec{a}, \vec{b}, \vec{c} = \frac{1}{6} |[\vec{a} \ \vec{b} \ \vec{c}]|$$

- (f) Volume of prism on a triangular base with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{2} |[\vec{a} \ \vec{b} \ \vec{c}]|$

Lagrange's identity :

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = \begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{a} \cdot \bar{d} \\ \bar{b} \cdot \bar{c} & \bar{b} \cdot \bar{d} \end{vmatrix} = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c})$$

Reciprocal system of vectors : If $\bar{a}, \bar{b}, \bar{c}$ be any three non coplanar vectors so that

$[\bar{a} \ \bar{b} \ \bar{c}] \neq 0$ then the three vectors $\bar{a}' \bar{b}' \bar{c}'$ defined by the

equations $\bar{a}' = \frac{\bar{b} \times \bar{c}}{[\bar{a} \ \bar{b} \ \bar{c}]}$, $\bar{b}' = \frac{\bar{c} \times \bar{a}}{[\bar{a} \ \bar{b} \ \bar{c}]}$, $\bar{c}' = \frac{\bar{a} \times \bar{b}}{[\bar{a} \ \bar{b} \ \bar{c}]}$ are called

the reciprocal system of vectors to the given vectors $\bar{a}, \bar{b}, \bar{c}$.

Relation between A.M., G.M. and H.M.
A.M. \geq G.M. \geq H.M.

Equality sign holds only when all the observations in the series are same.

Relationship between mean, mode and median :

(i) In symmetrical distribution
Mean = Mode = Median

(ii) In skew (moderately symmetrical) distribution
Mode = 3 median - 2 mean

Mean deviation for ungrouped data

$$M.D.(\bar{x}) = \frac{\sum |x_i - \bar{x}|}{n}, \quad M.D.(M) = \frac{\sum |x_i - M|}{n}$$

Mean deviation for grouped data

$$M.D.(\bar{x}) = \frac{\sum f_i |x_i - \bar{x}|}{N}, \quad M.D.(M) = \frac{\sum f_i |x_i - M|}{N}$$

where $N = \sum f_i$

Variance and standard deviation for ungrouped data

$$\sigma^2 = \frac{1}{n} \sum (x_i - \bar{x})^2, \quad \sigma = \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2}$$

Variance and standard deviation of a discrete frequency distribution

$$\sigma^2 = \frac{1}{n} \sum f_i (x_i - \bar{x})^2, \quad \sigma = \sqrt{\frac{1}{n} \sum f_i (x_i - \bar{x})^2}$$

Variance and standard deviation of a continuous frequency distribution

$$\sigma^2 = \frac{1}{n} \sum f_i (x_i - \bar{x})^2, \quad \sigma = \sqrt{\frac{1}{N} \sum f_i x_i^2 - (\sum f_i x_i)^2}$$

Coefficient of variation (C.V.) = $\frac{\sigma}{\bar{x}} \times 100$, $\bar{x} \neq 0$

For series with equal means, the series with lesser standard deviation is more consistent or less scattered.

Methods of solving a first order first degree differential equation :

(a) Differential equation of the form

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = f(x) \Rightarrow dy = f(x) dx$$

Integrating both sides we obtain

$$\int dy = \int f(x) dx + c \quad \text{or} \quad y = \int f(x) dx + c$$

(b) Differential equation of the form $\frac{dy}{dx} = f(x) g(y)$

$$\frac{dy}{dx} = f(x) g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x) dx + c$$

(c) Differential equation of the form of $\frac{dy}{dx} = f(ax + by + c)$:

To solve this type of differential equations, we put

$$ax + by + c = v \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{b} \left(\frac{dv}{dx} - a \right)$$

$$\therefore \frac{dv}{a + b f(v)} = dx$$

So solution is by integrating $\int \frac{dv}{a + b f(v)} = \int dx$

(d) Differential Equation of homogeneous type :

An equation in x and y is said to be homogeneous if it

can be put in the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ where $f(x, y)$ and $g(x, y)$ are both homogeneous functions of the same degree in x & y .

So to solve the homogeneous differential equation

$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, substitute $y = vx$ and so $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\text{Thus} \quad v + x \frac{dv}{dx} = f(v) \Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v}$$

Therefore solution is $\int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c$

Linear differential equations :

$$\frac{dy}{dx} + Py = Q \quad \dots\dots (1)$$

Where P and Q are either constants or functions of x .

Multiplying both sides of (1) by $e^{\int P dx}$, we get

$$e^{\int P dx} \left(\frac{dy}{dx} + Py \right) = Q e^{\int P dx}$$

On integrating both sides with respect to x we get

$$y e^{\int P dx} = \int Q e^{\int P dx} + c$$

which is the required solution, where c is the constant and

$e^{\int P dx}$ is called the integration factor.

